

Labelled sequent calculi for indexed modal logics

Eugenio Orlandelli



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Eugenio Orlandelli

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Chapter 1

Introduction

This work, which is a revised version of [Orlandelli, 2014], is a proof-theoretic study of quantified modal logics (QMLs) in the context of indexed modalities and transition semantics. We hope in this way to dispel some 'locus communis' on the intractability of quantified modal logics. The core of our work is the introduction and the systematic study of labelled sequent calculi for Indexed Modal Logics (IMLs).

An IML is defined as the set of indexed modal formulas that are valid on a class of transition frames. Axiomatic systems for IMLs have been introduced in [Corsi, 2009] and studied in [Orlandelli, 2019]. Nevertheless, save for some of the simpler calculi, there exists no proof that such systems are complete with respect to the intended class of transition frames. This is an instance of a general problem for QMLs: completeness results for QMLs are extremely difficult to find, and in most cases the quantified extensions of a complete propositional modal logic are not complete with respect to the intended semantics [Shehtman and Skvortsov, 1990; Garson, 1991; Cresswell, 1995, 2001; Corsi, 2002; Goldblatt, 2011].

Im modal logics the incompleteness phenomenon is widespread at the propositional level and it becomes even worst at the predicative level, see [Shehtman, 2006]. Even if we limit our attention to the quantified extensions of those propositional modal logics (PMLs) which are complete and canonical, we find that most of them are incomplete. Of course the analysis needs to be more precise be-

cause 'quantified extensions' can mean different things – e.g., there are logics with or without classical quantification, with or without the Barcan formulas, with or without the Ghilardi formula and with or without the necessity of identity. In general, given a PML **S** that is complete with respect to a given class of frames, two kinds of incompleteness phenomenons may arise for its quantified extensions: there are *incompletable* logics such as the non recursively axiomatizable **Q.GL** [Montagna, 1984] and *completable* logics such as **Q.S4M** where, in order to obtain a complete system, we need to add some *de re* axiom that regulates the interaction between modalities and the first-order machinery [Cresswell, 2001].

We are interested in completable logics: we aim at extending to the predicate level a wide range of completeness results established at the propositional level. In doing so, we want to solve the following open problem for IMLs:

> how should we define a modular family of proof systems that characterize quantified extensions of a wide class of complete PMLs?

This will be done by introducing labelled sequent calculi for IMLs which behave extremely well from a proof-theoretic point of view – all the structural rules of inference are admissible and all rules are invertible – and characterize the valid formulas of quantified extensions of any first-order definable PML.

These results show that labelled sequents are better behaved than axiomatic systems with respects to QMLs: for any completable – but incomplete – axiomatic system for a first-order definable QML, we can define a labelled sequent calculus that proves all theorems of the completion of the axiomatic system. To illustrate, in [Cresswell, 1995] it has been shown that the formula:

$$\Diamond(\forall x(A \to \Box A) \wedge \Box\neg\forall x A) \wedge \Diamond\forall x(A \vee \Box A) \wedge \forall x(\Diamond A \to \Box A) \quad (1.1)$$

is consistent in the axiomatic system **Q.2.BF** (i.e., Cresswell's system **KG1+BF**), but it is unsatisfiable in the class $\mathscr{C}^{2,BF}$ of all Kripke frames for that logic. This proves the incompleteness of **Q.2.BF**, which is completable by adding some presently unknown axiom. On the other

hand, our modular completeness theorem entails that the labelled sequent calculus for $\mathscr{C}^{2,BF}$ is complete with respect to it. As we will show in Figure 5.1, the negation of the formula 1.1 is derivable in the labelled calculus for $\mathscr{C}^{2,BF}$.

Before going into the complexities of IMLs, we briefly review the main tenets of the labelled sequent calculi for PMLs, see [Negri, 2005; Negri and von Plato, 2011].[1] Let a PML be the set of propositional modal formulas that are valid on a class of frames $\mathscr{F}^p = \langle \mathscr{W}, \mathscr{R} \rangle$ such that $\mathscr{F}^p$ is defined by a set of conditions on $\mathscr{R}$. The key idea of labelled calculi is that, by internalizing the possible-world semantics in the proof theory, we can talk about possible-world frames in proof-theoretic terms and, therefore, we can define a modular family of calculi with well-behaved structural properties. The internalization is obtained by extending the language with a set of world labels $w, v, \ldots$ This allows us to replace propositional modal formulas ϕ with labelled ones $w : \phi$, where $w : \phi$ intuitively means that 'world w forces ϕ' (ϕ is true at w). Moreover, it allows us to extend the language with relational atoms $w \mathscr{R} v$, where $w \mathscr{R} v$ intuitively means that 'world v is accessible from w'. We can thus extend the cut- and contraction-free sequent calculus **G3cp** with well-behaved rules for the modal operators. These rules internalize the semantical explanations of the forcing relation for modal formulas. To illustrate, let's consider the right-to-left arrow of the forcing condition for $\Box\phi$:

$$(\text{for all } v \in \mathscr{W},\ w \mathscr{R} v \text{ implies } \Vdash_v \phi) \text{ only if } \Vdash_w \Box\phi$$

This condition is captured by the following rule (where v cannot occur in the conclusion):

[1] One of the main goals of sequent calculi for PMLs is to define calculi that allow us to find derivations and to analyze the structural properties of derivations in a PMLs. Traditional sequent systems for PMLs fail to satisfy the required properties in many important cases, as it is witnessed by the difficulties in defining a cut-free system for **S5**. To overcome this limitation, various extensions of traditional sequent systems have been introduced, e.g., display logic [Belnap, 1982; Wansing, 1998], hyper(tree)- sequent calculi [Avron, 1991; Poggiolesi, 2011], and labelled sequent calculi [Negri, 2005; Negri and von Plato, 2011]. See [Negri, 2011; Poggiolesi, 2011] for comparisons of these approaches. Here we will deal with labelled calculi because they are better suited for our purposes.

$$\frac{w\mathcal{R}v, \Gamma \Rightarrow \Delta, v : \phi}{\Gamma \Rightarrow \Delta, w : \Box\phi} R\Box$$

Next, we have to internalize the semantic conditions on $\mathscr{R}$ that define the PML under consideration. This will be done by adding rules operating on relational atoms. If we were to do so by adding a condition on $\mathscr{R}$ directly as a nonlogical axiom, we would lose the admissibility of the structural rules because 'the *Hauptsatz* fails for systems with proper axioms' [Girard, 1987, p. 125]. Labelled sequent calculi overcome this difficulty by using the method of axioms as rules. This powerful technique produces a system that is equivalent to one with proper axioms without loosing the structural properties of the basic calculus. The method of axioms as rules has been introduced in [Negri and von Plato, 1998] for universal axioms, it has been extended in [Negri, 2003] to geometric ones,[2] and it has been shown to cover all first-order axioms in [Dyckhoff and Negri, 2015] (see also [Negri, 2014] for an alternative way of going beyond geometric axioms). By applying this methodology to conditions on $\mathscr{R}$, we can define a well-behaved sequent calculus for any first-order definable PML. To illustrate, the modal axiom T, which corresponds to reflexive frames, is captured by the following left nonlogical rule:

$$\frac{w\mathcal{R}w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} T$$

which, if read bottom-up, says that $w\mathcal{R}w$ holds for each label w.

Labelled sequent calculi work equally well for IMLs and, therefore, allow us to give a proof-theoretic characterization of a wide class of semantically defined logics. We conclude this introduction by recalling the fundamentals of the language and the semantics of IMLs, and by sketching how we will internalize transition semantics in labelled sequent calculi.

The main difference between the indexed language and the language of QMLs is thate propositional modal operators $\Box$ and $\Diamond$ are replaced by operators that are indexed by sets of pairs of terms and variables : $|^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|$ and $\langle {}^{t_1}_{x_1} \dots {}^{t_m}_{x_m} \rangle$. In this way we can define an indexed modal formula as

[2] A geometric formula is a formula of shape $\forall \vec{x}(A \to B)$ where neither A nor B contains $\to$ or $\forall$.

$$|^{t_1}_{x_1} \dots ^{t_n}_{x_n}|A \tag{1.2}$$

or

$$\langle^{t_1}_{x_1} \dots ^{t_m}_{x_m}\rangle A \tag{1.3}$$

where, in both cases, the variables occurring free in A must be among the variables in the lower row of the indexed operator. This proviso is needed because in transition semantics the satisfaction clause of a modal formula is parametric on the free variables of the formula in the scope of its main operator. The formula in 1.2 can be read as:

> it is necessary for $t_1, \dots, t_n$ that (they satisfy) A,

and that in 1.3 as:

> it is possible for $t_1, \dots, t_n$ that (they satisfy) A.

A transition frame is a propositional modal frame augmented with a set of domains indexed by possible worlds D_w[3] and by a set of binary relations, which are denoted as $\mathcal{T}_{(w,v)}$, between the objects of the domains of pairs of worlds w, v.[4] A model over a transition frame is obtained by adding, for each possible world w, a local interpretation I_w of the signature over D_w. Assignments are local mappings σ_w from the variables to D_w. In general terms are not rigid designators: at every world their extension is defined (over the domain of that world) irrespectively of their extension in the other worlds. By $\sigma_w(t)$ we denote the extension of the term t in w. We can now define the forcing condition for indexed modal formulas as follows:

$$\begin{aligned} &\sigma_w \models_w |^{t_1}_{x_1} \dots ^{t_n}_{x_n}|A \quad \text{iff} \quad \text{for all } v \text{ s.t. } w\mathcal{R}v \text{ and} \\ &\text{all } \sigma_v \text{ s.t. } \sigma_w(t_{i\leq n})\mathcal{T}_{(w,v)}\sigma_v(x_{i\leq n}), \qquad \sigma_v \models_v A \end{aligned} \tag{1.4}$$

which informally means that the formula $|^{t_1}_{x_1} \dots ^{t_n}_{x_n}|A$ is true at w of the tuple of objects $a_1, \dots, a_n$ (the w-extension of the t_is) whenever, in every accessible world v , the formula A is true of every tuple of

[3] Although we will work with transition frames with double domains, in this introduction we deal with single domains for the sake of simplicity.

[4] The transition relation can be seen as a modal version of the counterpart relation introduced by Lewis [1968] to give a fist-order theory of modality.

objects $b_1, \ldots, b_n$ (taken from D_v) such that each b_i is a counterpart of a_i. Observe that an indexed modal operator quantifies not on all accessible worlds, but only on those accessible worlds where there are counterparts of the objects we are talking about.

Clause 1.4 is equivalent to that of Kripke semantics with respect to closed formulas in the scope of an operator that is indexed by the empty set of (pairs of) terms. But it is more general with respect to open formulas in the scope of an indexed modal operator. Thus, it grants us a better control over the interaction of the first-order machinery (quantifiers, substitutions and identity) with modalities. As a consequence, many formulas that are valid on every Kripke frame correspond to particular conditions on the transition relations, and therefore transition semantics has a richer correspondence theory. By an IML we mean the set of indexed formulas that are valid on a class of transition frames that is defined by a set of conditions on the accessibility and on the transition relation.

In order to internalize transition semantics into **G3**-style calculi, we have to apply labels not only to formulas, but also to terms, which become expressions like t^w. This is needed because both formulas and terms have a locally defined extension at each world. Moreover, we have to extend the language not only relational formulas, but also transitional formulas like $t^w \mathcal{T} s^v$, which informally means that 'the extension of s in v is a counterpart of the extension of t in w'. We can now define the rules for the indexed modal operators as semantical explanations of the forcing conditions. To illustrate, from the right-to-left implication of clause 1.4, we obtain the following right rule (where neither v nor the x_i^vs can occur in the conclusion):

$$\frac{w \mathcal{R} v, t_1^w \mathcal{T} x_1^v, \ldots, t_n^w \mathcal{T} x_n^v, \Gamma \Rightarrow \Delta, v : A}{\Gamma \Rightarrow \Delta, w : |_{x_1}^{t_1^w} \ldots {}_{x_n}^{t_n^w} | A} \; R|_{\vec{x}}^{\vec{t}}|$$

By having internalized transition semantics, we immediately have a sequent calculus for the minimal IML. This calculus has good structural properties because the addition of modal rules doesn't affect the structural properties of the underlying calculus. Furthermore, by applying the method of axioms as rules, we can define a sequent calculus for any class of transition frames that is definable by means of universal and geometric properties of the accessibility and of the transition relations. To illustrate, let's consider all transition frames

where the transition relation is surjective:

> for all $w, v \in \mathscr{W}$ and for all $b \in D_v$, if $w\mathscr{R}v$, then there is an $a \in D_w$ such that $a\mathscr{T}_{(w,v)}b$.

Surjectivity corresponds to the (indexed) Barcan Formula (BF):

$$\forall x|y_1,\dots y_n\, x|A \rightarrow |y_1 \dots y_n|\forall x A$$

The labelled calculus for such class of transition frames is obtained by adding the following rule (where x^w cannot occur in the conclusion):

$$\frac{x^w \mathcal{T} t^v, w\mathcal{R}v, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, \Gamma \Rightarrow \Delta}\, BF$$

which, informally, means that '$w\mathscr{R}v$ implies that each member of D_v (i.e. the extension of t in v) is a counterpart of some member of D_w': the transition relation is surjective as desired.

For each class of transition frames that is defined by a set of first-order conditions we will introduce a calculus where weakening and contraction are height-preserving admissible and where cut is admissible. Moreover, as we will show, these calculi give a proof-theoretic characterization of the IML under consideration. Furthermore, we will not need to give a separate proof of completeness for each IMLs, but we will give a single modular proof that covers all the different cases.

To sum up, our proof-theoretic study will allow us to dissipate some of the main problems of QMLs, which are concisely stated in the following passage:

> [I]n many ways modal predicate logic is in the state today that modal propositional logic was itself in the late sixties and early seventies. We have a few results about particular systems, we lack results about other particular systems, and we have practically no general results at all about completeness in modal predicate logics. [Cresswell, 2001, p. 178]

Synopsis. In Chapter 2, we introduce the indexed modal language, and explain its novelties w.r.t. the ordinary language of QMLs. We will give particular attention to the notions of bound variables and of substitution of terms for variables. In Chapter 3, we present transition semantics based on double domains transition frames. We will also explain why transition semantics had to be coupled with the indexed modal language. We will also present the semantic conditions that define many important IMLs. We stress that our main objective in the first two chapters is not that of justifying the introduction of IMLs, for which the reader is referred to [Corsi, 2009; Corsi and Orlandelli, 2013; Orlandelli, 2019], but that of introducing all the notions that will be used in our proof-theoretic study of IMLs. For the same reason, we will not be interested in showing possible applications of IMLs, see [Corsi and Orlandelli, 2013; Corsi and Tassi, 2014], but only in showing that IMLs provide a good theoretical framework for studying quantified modal logics.

In Chapter 4, which is the core of this work, we introduce labelled sequent calculi for IMLs and we study in great detail their structural properties. We will be able to introduce a sequent calculus for each semantic class of transition frames we have introduced in Chapter 3. As we will show, in all our calculi the structural rules of weakening and contraction are height-preserving admissible, all rules are height-preserving invertible, and the structural rule of cut is admissible.

Finally, in Chapter 5, we prove, in a completely modular way, that each calculus is sound and complete with respect to the appropriate class of transition frames. We will also prove that labelled sequent calculi are stronger than axiomatic systems in that if a formula A is a theorem of an axiomatic system for a quantified modal logic, then it is provable in the labelled sequent calculus for that logic, but the converse implication does not hold in general: we have completed many incomplete axiomatic systems.

Chapter 2

Indexed Modal Language

This chapter presents the indexed modal language, introduced in [Corsi, 2009], which is based on changing the syntax of modal operators and of modal formulas by introducing indexed operators, i.e. operators that are parametrized by sets of terms. This will give us a fine-grained control of free variables and substitutions in modal contexts. With indexed operators we have formulas such as

$$|^{t_1}_{x} \ldots ^{t_n}_{x_n} | A(x_1, \ldots, x_n)$$

whose intended reading is

> It is necessary for $t_1, \ldots, t_n$ that they are in relation $A(x_1, \ldots, x_n)$.

The chapter proceeds as follows. We begin by introducing the language $\mathscr{L}^{\boxdot}$ of indexed modal logics. Then, in Section 2, we introduce the notion of bound variables, where a variable may be bound by a quantifier or by an indexed operator. Finally, in Section 3, we define substitutions and prove some properties thereof.

2.1 Language

Definition 2.1 (Language)**.** The *language* $\mathscr{L}^{\boxdot}$ consists of:

- A *signature*:

  - For every $n \in \mathbb{N}$, an at most denumerable set, REL, of n-ary relation symbols $R_1^n, R_2^n, \ldots$
  - An at most denumerable set, CON, of individual constants $c_1, c_2, \ldots$
  - For every $n \in \mathbb{N} - \{0\}$, an at most denumerable set, FUN, of n-ary function symbols $f_1^n, f_2^n, \ldots$
  - A binary relational constant $\doteq$ (equality).

- A (non-empty) denumerable set, VAR, of (individual) *variables* $x_1, x_2, \ldots$
- The *logical symbols*, or *logical operators*, and their respective arity:
  - The propositional *connectives*: the 0-ary $\bot$ (falsum), and the binary $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies);
  - The unary *quantifiers*: $\forall$ (forall), $\exists$ (exists);
  - The unary *indexed operators*: $|\cdot|$ (it is necessary), $\langle\cdot\rangle$ (it is possible), respectively called 'box' and 'diamond'.
- The *auxiliary symbols* '(', ')', ',' and '$\star$'. ■

Definition 2.2 (Terms)**.** The set of *terms*, $TERM$, is defined inductively as follows:

1. If $x_i \in VAR$, then $x_i \in TERM$;
2. If $c_i \in CON$, then $c_i \in TERM$;
3. If $f_i^n \in FUN$ and $t_1, \ldots, t_n \in TERM$, then $f_i^n(t_1, \ldots, t_n) \in TERM$. ■

We will usually omit the superscript indicating the arity of a function, and assume that the number of arguments shown is correct.

Definition 2.3. Given a term t, the set $VAR(t)$ of the *variables occurring in t* is defined inductively by:

1. $VAR(x_i) = \{x_i\}$;
2. $VAR(c_i) = \emptyset$;

3. $VAR(f(t_1,\ldots,t_n)) = VAR(t_1) \cup \ldots \cup VAR(t_n)$. ■

Given a tuple $\vec{t}$ of terms, $VAR(\vec{t}) = \bigcup\{VAR(t_i) : t_i \in \vec{t}\}$.
By a *closed term* we mean any term t such that $VAR(t) = \emptyset$, and by an *open term* we mean a term that is not closed.

Definition 2.4 (Formulas)**.** We are now going to define, by a simultaneous induction, the sets (a) $FORM$ of *formulas* and the set (b) $FV(A)$ of the *free variables of a formula A*.

1. (a) If $R_i^n \in REL$ and $t_1,\ldots,t_n \in TERM$, then $R_i^n(t_1,\ldots,t_n) \in FORM$;
   (b) $FV(R_i^n(t_1,\ldots,t_n)) = VAR(t_1) \cup \ldots \cup VAR(t_n)$;
2. (a) $\bot \in FORM$;
   (b) $FV(\bot) = \emptyset$;
3. (a) If $A, B \in FORM$, then $(A \circ B) \in FORM$, where $\circ \in \{\wedge, \vee, \rightarrow\}$;
   (b) $FV((A \circ B)) = FV(A) \cup FV(B)$;
4. (a) If $A \in FORM$ and $x \in VAR$, then $(QxA) \in FORM$, where $Q \in \{\forall, \exists\}$;
   (b) $FV((QxA)) = FV(A) - \{x\}$;
5. (a) If $A \in FORM$, $FV(A) \subseteq \{x_1,\ldots,x_n\}$, and $t_1,\ldots,t_n \in TERM$, then $(|^{t_1}_{x_1}\ldots^{t_n}_{x_n}|A) \in FORM$, and $(\langle^{t_1}_{x_1}\ldots^{t_n}_{x_n}\rangle A) \in FORM$;
   (b) $FV((|^{t_1}_{x_1}\ldots^{t_n}_{x_n}|A)) = FV((\langle^{t_1}_{x_1}\ldots^{t_n}_{x_n}\rangle A)) = VAR(t_1) \cup \ldots \cup VAR(t_n)$.

■

Definition 2.5. The set $FORM^{=}$ is defined as the set $FORM$ with the addition of:

6. (a) If $t, s \in TERM$, then $t \doteq s \in FORM^{=}$;
   (b) $FV(t \doteq s) = FV(t) \cup FV(s)$. ■

In practice, parentheses can be omitted following the usual conventions. As we did for functions, we will usually omit the superscript indicating the arity of a relation. Whenever the distinction doesn't matter, by '*formulas*' we will mean both $FORM$ and $FORM^{=}$. The set $ATOM$ of *atomic formulas* is the set of all formulas that do

not contain logical symbols. By a *closed formula*, or *sentence*, we mean any formula A such that $FV(A) = \emptyset$.

We wil use $\neg A$ as a shorthand for $A \to \bot$, $A \leftrightarrow B$ for $(A \to B) \wedge (B \to A)$, and $s \neq t$ for $\neg(s \doteq t)$. We introduce the following notational conventions for indexed operators:

- The formula $|x_1, \ldots x_n|A$ stands for $|^{x_1}_{x_1} \ldots ^{x_n}_{x_n}|A$.
- When the index is the empty sequence, we write $|\star|A$ and $\langle\star\rangle A$ (this may happen only if A is a sentence).
- In a modal formula $|^{t_1}_{x_1} \ldots ^{t_n}_{x_1}|A$, the tuple $t_1, \ldots, t_n$ is called the *numerator* of that formula, and the tuple $x_1, \ldots, x_n$ is called the *denominator* of that formula.

Observe that he free variables of an indexed formula are all and only the variables occurring inside its numerator, thus, in a sense that will become clearer later on, an indexed operator 'seals off' the formula occurring in its scope. As a matter of fact, the variables occurring in the denominator of an indexed operator bound all the ones occurring in the formula in its scope. This will allow us to introduce a fine-grained treatment of substitutions in modal contexts, and it will be of some importance for the indexed sequent calculi that we will introduce in Chapter 3.

We will use the following *metavariables* (all possibly with subscripts taken from $\mathbb{N} - \{0\}$):

- x, y, z for variables, and $\vec{x}, \vec{y}, \vec{z}$ for tuples of variables;
- a, b, c for individual constants, and $\vec{a}, \vec{b}, \vec{c}$ for tuples of constants;
- f, g for function symbols;
- r, s, t for terms, and $\vec{r}, \vec{s}, \vec{t}$ for tuples of terms;
- A, B, C for formulas (possibly containing equality), and $A(\vec{x})$ for a formula whose free variables are among $\vec{x}$;
- P, Q for atomic formulas (possibly containing equality), and $P(\vec{x})$ for an atomic formula whose free variables are among $\vec{x}$;

- p, q for 0-ary relation symbols, i.e. propositional atoms.

We will also denote logical operators by means of:

- '$\circ$' for binary propositional operators,
- 'Q' for the quantifiers,
- '$\boxdot$' for indexed operators.

Given a formula A, by a *subformula* of A we will mean any member of the construction tree of A. More precisely:

Definition 2.6 (Subformulas)**.** For any formula A the set $SF(A)$ of its *subformulas* is defined inductively by:

1. $SF(\bot) = \{\bot\}$;
2. $SF(R_i^n(t_1, \ldots, t_n)) = \{R_i^n(t_1, \ldots, t_n)\}$;
3. $SF(B \circ C) = SF(B) \cup SF(C) \cup \{B \circ C\}$;
4. $SF(QxB) = SF(B) \cup \{QxB\}$;
5. $SF(\boxdot B) = SF(B) \cup \{\boxdot B\}$. ■

In order to prove properties of terms and formulas, we will reason inductively on their height that are defined as follows.

Definition 2.7 (Term-height)**.** The *term-height of a term* t, $h(t)$, is defined inductively as follows:

1. $h(x) = h(c) = 0$;
2. $h(f(t_1, \ldots, t_n)) = max\{h(t_1), \ldots, h(t_n)\} + 1$. ■

Definition 2.8 (Formula-height)**.** The *formula-height of a formula* A, $h(A)$, is defined inductively as follows:

1. $h(\bot) = h(R_i^n(t_1, \ldots, t_n)) = h(s \doteq t) = 0$;
2. $h(Qx_iB) = h(\boxdot B) = h(B) + 1$;
3. $h(B \circ C) = max\{h(B), h(C)\} + 1$. ■

2.2 Bound Variables

Both quantifiers and indexed operators are devices for anaphora: they create anaphoric linkages between variables that, in this process, are deprived of their referential role. To illustrate, the formula

$$|^{c}_{x}|\exists y P(x,y) \tag{2.1}$$

may be read as

$$c \text{ is necessarily } P\text{-ing something.}^{1}$$

Given that anaphoric linkages are usually expressed in first-order logic by bound variables, in indexed modal logics we have two different kinds of bound variables: variables – e.g., y in 2.1 – bound by a quantifier and variables – e.g., x in 2.1 – bound by an indexed modal operator. We will call them *q-bound* and *m-bound* variables, respectively. Although the notion of an m-bound variable is new, it can easily be understood by analogy with the well-known notion of a q-bound variable. Given a formula

$$|^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|A \tag{2.2}$$

the variables $x_1,\dots,x_n$ occurring in the denominator of its indexed operator bound all occurrences of $x_1,\dots x_n$ in its subformula A. The main difference is that whereas q-bound variables have their anaphoric antecedent in a quantified expression Qx, m-bound variables have their anaphoric antecedent in a term occurring in the numerator of an indexed operator. To illustrate, in 2.2, each occurrences of $x_{i(\leq n)}$ in A has its anaphoric antecedent in the term $t_{i(\leq n)}$ occurring in the numerator of the indexed operator.

In first-oder logic the purely anaphoric role of bound variables is captured by the fact that α-congruent formulas, i.e. formulas that are identical modulo a renaming of q-bound variables, may be identified: they are proof-theoretically inter-derivable and semantically co-satisfiable. As we will see, the same property holds for indexed formulas that are identical modulo a renaming of some of its m-bound variables. In this case we will talk of *i-congruent formulas*.

[1] Here we assume that $P(x,y)$ is a transitive verb, in general the suggested reading of a formula $|^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|R^n$ is 'it is necessary for $t_1,\dots,t_n$ that they are in relation R^n'.

Observe that our definition of formulas entails that no variable occurs free in A whenever A is in the scope of an indexed operator.[2] This is one motivation for our claim that an indexed operator seals off the formula in its scope.

These considerations on α- and i-congruent formulas suggest that the only role of variables occurring in a formula A is that of expressing anaphoric linkages, unless they occur free in that formula. Thus, as long as we don't lose track of these linkages, we may dispense with bound variables. One way of dispensing with them is by replacing formulas with skeletons, see [Gabbay et al., 2009, p. 83-84], where the skeleton of a formula A is the expression that we obtain by replacing every bound occurrence of a variable in a formula A with a new expression that is graphically connected to its anaphoric antecedent. We replace q-bound variables with • and connect 'from above' each • with its anaphoric antecedent. Moreover, we replace m-bound variables with $*$ and connect 'from below' each occurrence of $*$ inside the scope of the operator with its corresponding occurrence in the denominator.[3] This allows us to display the difference between the two kinds of anaphoric linkages. Formally:

Definition 2.9 (Skeleton)**.** For any formula A, its *skeleton* $\ulcorner A \urcorner$ is defined inductively as follows:

1. $\ulcorner A \urcorner = A$, for A atomic or $\bot$;
2. $\ulcorner (A \circ B) \urcorner = \ulcorner A \urcorner \circ \ulcorner B \urcorner$;
3. $\ulcorner Qx A \urcorner$ is obtained from $Qx \ulcorner A \urcorner$ by replacing every occurrence of x with • and connecting it with the displayed occurrence of Q;
4. $\ulcorner |^{t_1}_{x_1} \dots ^{t_n}_{x_n} | A \urcorner$ is obtained from $|^{t_1}_{x_1} \dots ^{t_n}_{x_n} | \ulcorner A \urcorner$ by replacing every occurrence of each x_i in the denominator and in $\ulcorner A \urcorner$ with $*$ and, then, connecting each occurrence of $*$ in $\ulcorner A \urcorner$ with its occurrence inside the denominator;

[2] To wit, the free occurrences of variables in $|^{\vec{t}}_{\vec{x}}| A$ are all and only the occurrences of variables in the numerator $\vec{t}$, since all variables occurring in A are bound either by $|^{\vec{t}}_{\vec{x}}|$ or by some other binding operator in its scope.

[3] Which is anaphorically linked to the term (or • or $*$) in the corresponding position of the numerator.